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HOT, VIBRATING NEUTRON STARS

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ABSTRACT

Taking into account the dissipative effect of neutrino-producing reactions, we calculate the amount of thermal and vibrational energy that can be present in a neutron star at a time t after its creation in a supernova explosion.

The results are used to discuss the plausibility of a number of hypotheses advanced in recent years concerning possible observable properties of neutron stars.

I. INTRODUCTION

If a supernova explosion results from the gravitational collapse of a star's core into a neutron star, this neutron star is likely to begin its life with a large amount of thermal and vibrational energy. The star might conceivably be detectable if a substantial part of the available energy were converted into electromagnetic radiation.

Unfortunately, calculations show (Chiu and Salpeter 1964; Finzi 1964, 1965*a;* Bahcall and Wolf 1965) that most of the thermal energy of a hot, non-vibrating star is rapidly dissipated by neutrinos produced in the reactions (Urca reactions)

$$n + n \rightarrow n + p + e + \bar{\nu}\,, \tag{1}$$

$$n + p + e \rightarrow n + n + \nu\,. \tag{2}$$

The rate of decrease of the surface temperature with time can be derived by using the calculated rates of reactions (1) and (2) to deduce the rate of decrease of the core temperature and by using neutron-star models to relate the core temperature to the surface temperature. Neutron-star models are, unfortunately, strongly dependent on the assumptions made about nuclear forces. However, from the results of Tsuruta and Cameron (1966*a*, *b*), one should probably expect a surface luminosity of about 10^{35} ergs sec^{-1} for a neutron star a few hundred years old. Most of the photons emitted from the surface would be soft X-rays.

A star with an X-ray luminosity of 10^{35} ergs sec^{-1} could be detected at a distance of a few kiloparsecs, with only a moderate improvement of present techniques. However, the remnant of the supernova in Taurus, and probably the remnants of Tycho's supernova and the supernova in Cassiopeia as well (Friedman, Byram, and Chubb 1967; Friedman 1967), produce X-rays at a rate of a few times 10^{36} ergs sec^{-1}, substantially more than the 10^{35} ergs sec^{-1} suggested by the theoretical arguments mentioned above. At least in the case of the supernova in Taurus, we know directly from measurements of the X-ray spectrum and the angular size of the source, that the primary source of X-rays is not the black-body radiation of a neutron star. It would then be quite hard to establish the existence of a weak, black-body source against the background of the stronger source.

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Like the thermal energy of a hot, non-vibrating star, the vibrational energy of a cold, vibrating neutron star tends to be dissipated by the Urca reactions (Finzi 1965*b*). We shall see, however, that the amount of vibrational energy that could still be available a few weeks after the explosion would probably be consistent with the suggestion (Finzi 1965*b*) that the light of a supernova comes from the vibrational energy of the neutron star created in the catastrophic event.[1] Besides the approximate agreement between the vibrational energy that can be available in a neutron star and the radiative energy released by a supernova, there is perhaps another reason for assuming that the radiative energy originates from some starlike object left after the outburst; it has been pointed out by Colgate and White (1966) that the radiative energy produced in the sudden outburst will be converted into kinetic energy of the ejected shell before it is able to escape into interstellar space.

To complete the review of conceivable ways of detecting the collapsed body left at the center of a supernova, we should mention that acceleration of relativistic electrons seems to occur at the center of the Crab Nebula at the rate of about 10^{38} ergs sec^{-1} (Shklovskii 1966). Since the low-density plasma could hardly release this much energy, it is tempting to speculate that the energy is being pumped into the electrons by some invisible collapsed body. However, a neutron star almost 1000 years old definitely cannot release 10^{38} ergs sec^{-1} at the expense of either its thermal-energy content or its vibrations. If we want to stick to the idea that the 10^{38} ergs sec^{-1} come from the energy stored in a neutron star and also to remain within the limits of present theoretical ideas, we can only assume that the relativistic electrons are accelerated at the expense of the magnetic energy of the star. Large amounts of magnetic energy could be present in a neutron star (Woltjer 1964).

The rate of energy loss by Urca reactions seems, therefore, to be an important factor in determining the detectability of neutron stars. Meltzer and Thorne (1967) have suggested, however, that the results deduced for the two limiting cases studied so far (hot, non-vibrating stars and cold, vibrating stars) may not provide accurate information for the more general case of a hot, vibrating star. Specifically, the Urca reactions might steadily transform vibrational energy into thermal energy, and this might result in a significant increase of the central temperature and of the thermal emission from the surface.

The neutrino luminosity of a hot, vibrating neutron star has been calculated by Hansen (1966) and by Hansen and Tsuruta (1967): however, as we shall explain in § III, these authors have not calculated the rate of change of thermal energy or the rate of dissipation of vibrational energy. The rate of dissipation of vibrational energy is calculated in § V of the present paper. The rate of change of thermal energy is the difference between the rate of dissipation of vibrational energy and the neutrino luminosity.

In § VI we study the evolution of neutron stars that begin their lives with non-zero vibrational amplitudes and lose thermal and vibrational energy by the Urca reactions exclusively. We find that the ratio of the vibrational energy to the thermal energy rapidly approaches a constant value. For a particularly simple model considered in detail, this value is equal to 7; the vibrational energy is 3.5×10^{49} ergs year$^{-1/3}$.

In § VII we find that the rate of temperature decrease for a vibrating star is generally slower than for a non-vibrating star, as predicted by Meltzer and Thorne; also, the rate of vibrational damping is generally faster for a hot star than for a cold star. However, both these effects are sufficiently small that our previous conclusions concerning observable properties of neutron stars do not have to be modified. On the other hand, it is not inconceivable that neutron stars may lose most of either their thermal or their vibrational energy by some means other than the Urca process. The stars would then be described by the solution for one of the two limiting cases.

Suppose, for example, that there exists a mechanism for transforming the vibrational

[1] The total amount of radiant energy emitted by a type I supernova should be of the order of a few times 10^{49} ergs.

energy of a neutron star into the radiative energy of a supernova (Finzi 1965*b*). After this has taken place, the neutron star can be described by the solution for a hot, non-vibrating star.

Alternatively, if gravitational radiation exists in nature and it can be accurately estimated in the weak-field limit, Chau's results (1967) indicate that this radiation would very quickly damp the vibrations of a neutron star. In this case, too, the star could then be described by the solution for a hot, non-vibrating star.

On the other hand, suppose that there exists some heat-transfer mechanism, much more efficient than the conventional processes considered by Tsuruta and Cameron, that could carry heat rapidly from the core of a neutron star to the surface. The action of the Urca process could then be described by the solution for a cold, vibrating star. We show in § VII that the Urca process transforms only $\frac{3}{8}$ of the vibrational energy of a cold star into neutrino energy, while the balance is transformed into thermal energy, which is eventually emitted from the surface in the form of black-body radiation. The cold, vibrating star would thus be a strong X-ray emitter. This model is somewhat artificial, but it indicates that one cannot completely rule out the possibility that a neutron star a few hundred years old may have an X-ray luminosity of several times 10^{36} ergs $\sec^{-1}$.

Finally, we should mention that our assumption that the Urca processes are the most powerful neutrino-producing reactions is subject to two provisos: (*a*) Superfluidity in the proton gas is not important. A very large superfluid gap in the spectrum of the proton gas could reduce the rate of reactions (1) and (2) sufficiently to make other dissipative reactions more important (Ginzburg and Kirzhnits 1965; Ruderman and Festa 1966; Wolf 1966). In our present calculations we assume that the superfluid gap, if it exists, is not large enough to be important at the temperatures and vibrational amplitudes considered. (*b*) Negative pions are not present in the neutron star. If negative pions were present, thermal and vibrational energy could be lost in an extremely short time (Bahcall and Wolf 1965).

II. CHEMICAL EQUILIBRIUM IN NEUTRON-STAR MATTER

For densities less than 10^{15} gm cm^{-3}, it seems reasonable to picture neutron-star matter as consisting primarily of neutrons, together with a smaller number of protons and electrons. Very soon after the supernova explosion, the neutron star will cool off to temperatures less than 10^{10} ° K; at these temperatures, the neutron, proton, and electron gases are degenerate. The thermal conductivity of this degenerate matter is very high, and the star's core is thus likely to be isothermal. In this paper, we shall use T to designate the core temperature.

In a Fermi gas, the probability that an individual-particle state with energy W is occupied is given by $\{1 + \exp[(W - \mu)/kT]\}^{-1}$ where μ is the chemical potential. If one neglects interparticle interactions and assumes that the gas is highly degenerate, one can write the chemical potential of any fermion species i as the sum of its rest energy and Fermi energy, i.e.,

$$\mu_i = m_i c^2 + E_F(i) \,. \tag{3}$$

The strong interactions modify the proton and neutron chemical potentials; these modifications have been estimated by Wolf (1966). The corrections for the effects of strong interactions are still sufficiently uncertain, however, that it is not worthwhile to try to take them consistently into account in the present calculations.

The condition of chemical equilibrium among the neutrons, protons, and electrons can be expressed easily in terms of the chemical potentials:

$$\mu_n = \mu_p + \mu_e \,. \tag{4}$$

III. THE URCA PROCESSES

We consider neutrino emission by reactions (1) and (2) in a highly degenerate gas of neutrons, protons, and electrons. If the neutron star is not vibrating, then chemical equilibrium obtains, and most of the neutrons, protons, and electrons taking part in reactions (1) and (2) have energies that differ from their respective chemical potentials by amounts of the order of kT. The number of available states for each incoming or outgoing particle is then proportional to kT; the reaction rates consequently are equal to zero at zero temperature and increase rapidly with increasing temperature.

However, reactions (1) and (2) can also occur in a zero-temperature star that is vibrating: the chemical potentials of the non-relativistic, strongly interacting neutrons and protons do not depend on the density in the same way as the chemical potential of the relativistic, non-interacting electrons. Consequently, the equilibrium condition $\mu_n = \mu_p + \mu_e$ is generally not satisfied during a vibration. The result is that reaction (1) operates during a contraction, while reaction (2) operates during an expansion. These reactions will dissipate the vibrational energy of the star; a fraction of this energy will go into neutrinos and antineutrinos, while the balance will be transformed into thermal energy. Therefore, the same reactions (1) and (2) cool the star in one limiting case and heat the star in the other limiting case.

In the general case, we must follow the time evolution of both the vibrational amplitude and the core temperature. For this purpose, we derive two relations: the first relation gives the neutrino luminosity of a hot, vibrating neutron star and therefore the rate of decrease of the total vibrational and thermal energy; the second relation gives the net rate at which work is done against pressure forces and therefore the rate of loss of vibrational energy alone. Hansen and Tsuruta (Hansen 1966; Hansen and Tsuruta 1967) have calculated only the total neutrino luminosity and thus have no safe way to decide how the total loss is divided between the thermal and vibrational energy. Finzi (1965*b*) has calculated the rate of loss of vibrational energy for a zero-temperature vibrating star, but the method used in his paper does not apply directly to a hot, vibrating star.[2]

IV. NEUTRINO LUMINOSITY OF A HOT, VIBRATING NEUTRON STAR

Let v_0 be the volume occupied by a unit mass in a given slab of the neutron star at dynamical equilibrium, and let $v = v_0 + \delta v_t$ be the volume occupied at time t during a pulsation. The extent to which the slab is out of chemical equilibrium can be characterized by $\mu_n - \mu_p - \mu_e$, which, for small vibrations, is approximately equal to $(\partial\mu_n/\partial v - \partial\mu_p/\partial v - \partial\mu_e/\partial v)\delta v_t$. We introduce a dimensionless parameter u_t defined by the relation

$$u_t = \left(\frac{\partial\mu_n}{\partial v} - \frac{\partial\mu_p}{\partial v} - \frac{\partial\mu_e}{\partial v}\right)\frac{\delta v_t}{kT}. \tag{5}$$

We then let $L_\nu(u_t,T)$ be the neutrino luminosity of the unit mass.

It is convenient to express $L_\nu(u_t,T)$ in terms of $L_\nu(0,T)$, the neutrino luminosity per unit mass in the same slab when the star is not vibrating. Bahcall and Wolf (1965) express $L_\nu(0,T)$ as the sum of two equal terms representing the contributions of reactions (1) and (2); their expression for the contribution of reaction (1) is

$$L_\nu^{(1)}(0,T) = 64\pi^4\Omega G^2\hbar^{-1}\lambda\!\!\!^{-}{}_\pi^{-9}(C_V^2|M_V|^2 + 3C_A^2|M_A|^2)P\,, \tag{6a}$$

where the phase-space factor P is given by

$$P = 2^{-9}\pi^{-13}c^{-19}m_\pi^{-15}(m_n^*)^3 m_p^* P_F(p)P_F(e)^2(kT)^8 I(0)\,, \tag{6b}$$

and

$$I(0) = \int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_{-\infty}^{\infty}dx_3\int_{-\infty}^{\infty}dx_4\int_{-(x_1+x_2+x_3+x_4)}^{\infty}\left(\sum_{i=1}^{5}x_i\right)^3\prod_{i=1}^{5}(1+e^{x_i})^{-1}dx_5 = \frac{11513\pi^8}{120960}. \tag{6c}$$

In writing equation (6b), we have included a factor $(kT)^8$, which was left out of equation (31a) of Bahcall and Wolf (1965).

[2] We would like to correct two numerical errors in that paper. A factor of 3 is lacking in the expression for the vibrational energy W_T (eq. [31] of the present paper). There is a redundant factor of $\frac{1}{3}$ in

If we adopt the estimate of Bahcall and Wolf (1965) for the coefficient of P in equation (6a) and the estimate of Wolf (1966) for the coefficient of $I(0)$ in equation (6b), we arrive at the estimate

$$L_\nu^{(1)}(0,T) = (3 \times 10^5 \text{ ergs g}^{-1} \text{ sec}^{-1})(\rho_{\text{nucl}}/\rho)(T/10^9\,{}^\circ\text{ K})^8 \tag{7}$$

for the neutrino luminosity due to reaction (1) for the case $u_t = 0$. Here ρ_{nucl} is the density of nuclear matter, 3.7×10^{14} g cm^{-3}.

In order to consider the more general case, $u_t \neq 0$, we only have to modify the expression for I. None of the other factors depends on whether or not the star vibrates. The integration variables x_i appearing in equation (6c) were defined by Bahcall and Wolf to be equal to $\pm(W_i - \mu_i)/kT$, where W_i and μ_i are the energy and chemical potential of particle i. The index i runs from 1 to 5, the values 1 and 2 designating the two incoming neutrons in reaction (1) and the values 3, 4, and 5 designating the outgoing neutron, proton, and electron; the sign is $+$ for the incoming particles and $-$ for the outgoing particles. For the case in which the star does not vibrate, we have

$$\mu_1 + \mu_2 - \mu_3 - \mu_4 - \mu_5 = 0\,,$$

and we therefore find that the expression

$$\sum_{i=1}^{5} x_i = (W_1 + W_2 - W_3 - W_4 - W_5)/kT$$

represents the energy of the outgoing antineutrinos in units of kT. The integrand in the expression for $I(0)$ is thus proportional to the cube of the energy of the antineutrino.

To take the effect of small vibrations into account, we note that

$$\mu_1 + \mu_2 - \mu_3 - \mu_4 - \mu_5 = u_t kT\,,$$

so that the energy of the antineutrino, in units of kT, is given by

$$(W_1 + W_2 - W_3 - W_4 - W_5)/kT = u_t + \sum_{i=1}^{5} x_i\,.$$

The desired expression for $L_\nu^{(1)}(u_t,T)$ can thus be obtained simply by substituting for the quantity $I(0)$ in equation (6b) the quantity

$$I(u_t) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} dx_4 \int_{-(x_1+x_2+x_3+x_4+u_t)}^{\infty} \left(u_t + \sum_{i=1}^{5} x_i\right)^3 \prod_{i=1}^{5} (1 + e^{x_i})^{-1} dx_5\,. \tag{8}$$

The neutrino luminosity of neutron-star matter in a hot, vibrating star is then simply given by

$$L_\nu^{(1)}(u_t,T) = \frac{L_\nu^{(1)}(0,T)I(u_t)}{I(0)}\,. \tag{9}$$

In the expression for $I(u_t)$, four integrations can be performed analytically; one then finds that

$$L_\nu^{(1)}(u_t,T) = L_\nu^{(1)}(0,T)\frac{f(u_t)}{f(0)}\,, \tag{10a}$$

where

$$f(u_t) = \int_0^{\infty} \frac{y^3(y - u_t)^4}{1 + e^{y-u_t}}\,dy\,. \tag{10b}$$

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the expression for ΔE, the maximum energy available for an Urca reaction. The two errors partially compensate each other.

We are interested in the average of the neutrino luminosity over one vibrational cycle. If $u_t = u \cos \omega t$, we have

$$L_\nu(u,T) = 2L_\nu^{(1)}(0,T)\,\frac{F(u)}{F(0)}\,, \tag{11a}$$

where

$$F(u) = \frac{1}{\pi}\int_0^\pi d\theta \int_0^\infty \frac{y^3(y - u\cos\theta)^4}{e^{y-u\cos\theta}+1}\,dy \quad \text{and} \quad F(0) = \frac{127}{240}\,\pi^8 \simeq 5021\,. \tag{11b}$$

Also, we have introduced a factor 2 in equation (11a) to include the contribution of reaction (2). The quantity $F(u)/F(0)$ is plotted in Figure 1.

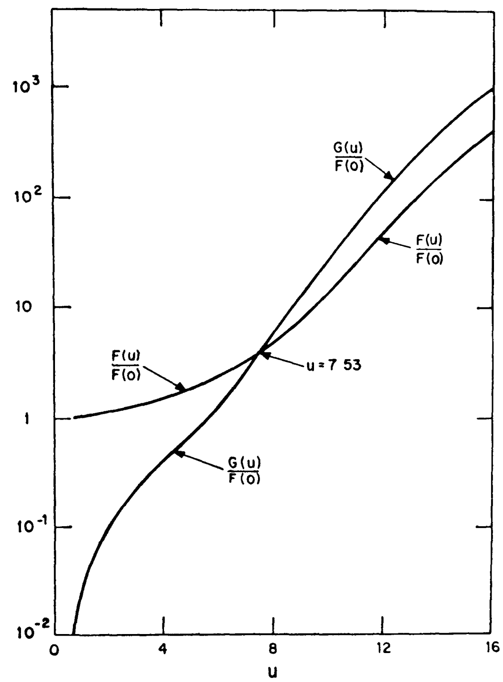

FIG. 1.—Functions $F(u)/F(0)$ and $G(u)/F(0)$

V. DAMPING OF VIBRATIONS BY THE URCA PROCESSES

The average rate of dissipation of vibrational energy per unit mass in a slab of neutron-star matter is given by

$$\frac{dw}{dt} = -\frac{1}{\tau}\int_0^\tau P(t)\,\frac{dv}{dt}\,dt\,, \tag{12}$$

where $P(t)$ is the pressure at time t and $\tau = 2\pi/\omega$ is the period of the vibration. We assume that δv_t, the time-dependent part of v, is given by

$$\delta v_t = \delta v \cos \omega t\,; \tag{13a}$$

we can write

$$P(t) = P_0 - \delta P \cos \omega t + \Pi(t)\,, \tag{13b}$$

where $\Pi(t)$ is a very small correction representing the change in pressure due to the occurrence of reactions (1) and (2). (We are assuming that no other dissipative mechanisms operate.)

Substituting equations (13a) and (13b) in equation (12), we obtain

$$\frac{dw}{dt} = \frac{\omega^2 \delta v}{2\pi}\int_0^\tau \sin \omega t\,\Pi(t)\,dt\,. \tag{14}$$

Integrating by parts, and noting that $\Pi(t)$ has to be accurately periodic, we find that

$$\frac{dw}{dt} = \frac{\delta v}{\tau} \int_0^\tau \cos \omega t \frac{d\Pi}{dt} dt . \tag{15}$$

We now have to calculate $d\Pi/dt$. Each single occurrence of reaction(1) in a unit mass of stellar matter destroys *one* neutron and adds *one* proton and *one* electron. The resulting change in pressure is

$$-\frac{\partial P}{\partial n_n} + \frac{\partial P}{\partial n_p} + \frac{\partial P}{\partial n_e} ,$$

where n_n, n_p, and n_e are the numbers of neutrons, protons, and electrons per unit mass. If the vibrations are small and kT is much less than the neutron Fermi energy, it is permissible to calculate $\partial P/\partial n_n$, $\partial P/\partial n_p$, and $\partial P/\partial n_e$ for zero temperature and chemical equilibrium. In the zero-temperature approximation, the pressure P and chemical potential μ_i satisfy the relation

$$\frac{\partial P}{\partial n_i} = -\frac{\partial \mu_i}{\partial v} ; \tag{16}$$

it follows that the decrease in pressure caused by a single occurrence of reaction (1) is given by

$$\Delta_1 P = \frac{\partial \mu_n}{\partial v} - \frac{\partial \mu_p}{\partial v} - \frac{\partial \mu_e}{\partial v} . \tag{17}$$

Within the approximation where $\partial P/\partial n_n$, $\partial P/\partial n_p$, and $\partial P/\partial n_e$ are calculated for zero temperature and chemical equilibrium, the pressure change due to a single occurrence of reaction (2) is equal and opposite to the pressure change caused by a single occurrence of reaction (1). Thus the rate of change of pressure caused by reactions (1) and (2) is given by

$$\frac{d\Pi}{dt} = [R^{(1)}(t) - R^{(2)}(t)]\left(\frac{\partial \mu_n}{\partial v} - \frac{\partial \mu_p}{\partial v} - \frac{\partial \mu_e}{\partial v}\right) , \tag{18}$$

where $R^{(1)}(t)$ and $R^{(2)}(t)$ are the rates of reactions (1) and (2) per unit mass.

We can easily deduce the expression for the rate $R^{(1)}(u_t,T)$ from the expression for the neutrino luminosity $L_\nu^{(1)}(u_t,T)$ derived in the previous section. We simply have to remove from the phase-space integral $I(u_t)$ a factor representing the antineutrino energy; in the notation used in equation (8), the antineutrino energy is $kT(u_t + \Sigma^5_{i=1} x_i)$. We thus conclude that

$$\frac{R^{(1)}(u_t,T)}{L_\nu^{(1)}(u_t,T)} = \frac{J(u_t)}{I(u_t)kT} \tag{19a}$$

or

$$\frac{R^{(1)}(u_t,T)}{L_\nu^{(1)}(0,T)} = \frac{J(u_t)}{I(0)kT} , \tag{19b}$$

where

$$J(u_t) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \int_{-\infty}^{\infty} dx_4 \int_{-(x_1+x_2+x_3+x_4+u_t)}^{\infty} \left(u_t + \sum_{i=1}^{5} x_i\right)^2 \prod_{i=1}^{5} (1 + e^{x_i})^{-1} dx_5 . \tag{19c}$$

We perform four integrations exactly and obtain the expression

$$\frac{R^{(1)}(u_t,T)}{L_\nu^{(1)}(0,T)} = \frac{1}{kT} \frac{g(u_t)}{F(0)} , \tag{20a}$$

where

$$g(u_t) = \int_0^\infty \frac{y^2(y - u_t)^4}{e^{y-u_t} + 1} dy . \tag{20b}$$

Using the fact that $R^{(2)}(u_t,T) = R^{(1)}(-u_t,T)$, we substitute equation (20a) in equation (18), to obtain

$$\frac{d\Pi}{dt} = \frac{L_\nu^{(1)}(0,T)}{kTF(0)}\left(\frac{\partial\mu_n}{\partial v} - \frac{\partial\mu_p}{\partial v} - \frac{\partial\mu_e}{\partial v}\right)[g(u_t) - g(-u_t)] . \tag{21}$$

Substitution of equation (21) in equation (15) yields the following expression for the rate of dissipation of vibrational energy:

$$\frac{dw}{dt} = \frac{L_\nu^{(1)}(0,T)}{kTF(0)}\left(\frac{\partial\mu_n}{\partial v} - \frac{\partial\mu_p}{\partial v} - \frac{\partial\mu_e}{\partial v}\right)\frac{\delta v}{\tau}\int_0^\tau \cos\omega t[g(u_t) - g(-u_t)]dt . \tag{22}$$

If we again assume that $u_t = u \cos \omega t$, define

$$G(u) = \frac{1}{\pi}\int_0^\pi d\theta u \cos\theta \int_0^\infty \frac{y^2(y - u\cos\theta)^4}{e^{y-u\cos\theta} + 1}dy , \tag{23}$$

and use equation (5) to relate δv and u, we finally obtain a convenient expression for the rate of dissipation of vibrational energy per unit mass:

$$\frac{dw}{dt} = 2L_\nu^{(1)}(0,T)\frac{G(u)}{F(0)} . \tag{24}$$

An expression for $L_\nu^{(1)}(0,T)$ is given in equation (7); the function $G(u)/F(0)$ is plotted in Figure 1.

VI. TIME EVOLUTION

a) Basic Equations

We can now write down the energy-balance relations that determine the vibrational amplitude and core temperature as functions of time, assuming that the Urca reactions are the only dissipative processes operating in the neutron star.

If we assume that the neutrons make the dominant contribution to the thermal energy W_T and that they form a perfect degenerate gas, we find that

$$W_T = \tfrac{1}{2}CT^2 , \tag{25}$$

where

$$C \approx [5 \times 10^{14} \text{ ergs g}^{-1}(10^9 \text{ °K})^{-2}]\int_0^M dM_r(\rho_{\text{nucl}}/\rho)^{2/3} . \tag{26}$$

The rate of change of thermal energy is the difference between the rate of dissipation of vibrational energy and the neutrino luminosity. From equations (11a), (25), and (26), we thus deduce that

$$CT\frac{dT}{dt} = 2\int_0^M dM_r L_\nu^{(1)}(0,T)\left[\frac{G(u) - F(u)}{F(0)}\right] . \tag{27}$$

If the star is executing small radial vibrations in a single mode with frequency ω, we can express the displacement δr of a mass point in terms of a dimensionless amplitude A and a function $\xi(r)$ uniquely defined by the condition to be equal to unity at the center. We thus write

$$\delta r(t) = rA\xi(r)\cos\omega t , \tag{28}$$

Within the usual linear treatment of stellar vibrations (see, e.g., Ledoux and Walraven 1958), the vibrational energy of the star can be expressed in the form

$$W = \tfrac{1}{2}BA^2 , \tag{29}$$

where

$$B = \int_0^R 4\pi r^2 dr \left[\Gamma P \left(3\xi + r \frac{d\xi}{dr} \right)^2 - 4GM_r \rho \xi^2 r^{-1} \right] \tag{30}$$

and Γ is the usual adiabatic coefficient $d \ln P/d \ln \rho$. In the simple case in which both Γ and ξ are independent of r, we find that

$$W = \tfrac{3}{2}(\Gamma - \tfrac{4}{3})\Omega A^2 \tag{31}$$

where Ω is the gravitational energy of the star. The rate of change of vibrational energy is, according to equations (24) and (29), given by

$$BA \frac{dA}{dt} = -2 \int_0^M dM_r L_\nu^{(1)}(0,T) \left[\frac{G(u)}{F(0)} \right]. \tag{32}$$

To first order, the variation in the specific volume v at a distance r from the center can be written as

$$\frac{\delta v_t}{v} = A \cos \omega t \left[3\xi(r) + r \frac{d\xi(r)}{dr} \right]. \tag{33}$$

and, using equation (5), we can write

$$u = \alpha(r) z\,, \tag{34}$$

where

$$\alpha(r) = \frac{v}{k} \left| \frac{d\mu_n}{dv} - \frac{d\mu_p}{dv} - \frac{d\mu_e}{dv} \right| \left| 3\xi + r \frac{d\xi}{dr} \right|, \tag{35}$$

$$z = A/T\,. \tag{36}$$

It is convenient to rewrite equations (27) and (32) using z and T as dependent variables. After some manipulation, we obtain

$$T^{-7} \frac{dT}{dt} = \int_0^M dM_r \left[\frac{2L_\nu^{(1)}(0,T)}{CT^8 F(0)} \right] [G(\alpha z) - F(\alpha z)]\,, \tag{37}$$

$$T^{-6} \frac{1}{z} \frac{dz}{dt} = \int_0^M dM_r \left[\frac{2L_\nu^{(1)}(0,T)}{CT^8 F(0)} \right] \left[F(\alpha z) - \left(1 + \frac{C}{Bz^2} \right) G(\alpha z) \right]. \tag{38}$$

b) Special Cases

Given the initial values z_i and T_i of the parameter z and temperature T, one can integrate equations (37) and (38) *numerically* to obtain z and T at any subsequent time. However, equations (37) and (38) can easily be integrated *analytically* for three special values of z_i.

Solution 1: $z_i = 0$, $T_i > 0$. If $z = 0$ at $t = 0$, then z will always remain zero. Equation (37) can then be integrated trivially, and one finds that

$$\frac{T}{T_i} = \left[1 + \frac{12t}{CT_i^2} \int_0^M dM_r L_\nu^{(1)}(0,T_i) \right]^{-1/6}. \tag{39}$$

This result applies to the case of a non-vibrating star, which was previously treated by Finzi (1965*a*) and by Bahcall and Wolf (1965).

Solution 2: $z_i = z_0$, $T_i > 0$, where z_c is the solution to the equation

$$\int_0^M dM_r \left[\frac{2L_\nu^{(1)}(0,T)}{CT^8 F(0)} \right] \left[F(\alpha z_c) - \left(1 + \frac{C}{Bz_c^2} \right) G(\alpha z_c) \right] = 0\,. \tag{40}$$

If $z = z_c$ at $t = 0$, then z will always be equal to z_c. We then obtain

$$\frac{T}{T_i} = \left[1 + \frac{12t}{BT_i^2 z_c^2} \int_0^M L_\nu^{(1)}(0,T_i) \frac{G(\alpha z_c)}{F(0)} dM_r\right]^{-1/6}. \tag{41a}$$

The vibrational amplitude is given by

$$A = z_c T . \tag{41b}$$

The constancy of z in time implies that the ratio between the vibrational and thermal energies has the constant value Bz_c^2/C.

Solution 3: $z_i = \infty$, $T_i > 0$. In this case the temperature initially increases while z decreases, rapidly approaching z_c. Except for extremely small values of t, this solution practically coincides with solution 2, with $T_i = \infty$. We then obtain

$$T = \left[\frac{12t}{Bz_c^2} \int_0^M \frac{L_\nu^{(1)}(0,T_i)}{T_i^8} \frac{G(\alpha z_c)}{F(0)} dM_r\right]^{-1/6}, \tag{42a}$$

$$A = z_c T . \tag{42b}$$

c) *Integrations in the General Case*

Equations (37) and (38) can be integrated numerically for arbitrary initial values of z. Dividing equation (37) by equation (38) and integrating, one can obtain T as a function of z:

$$\begin{aligned}\frac{T(z)}{T_i} = \exp \Big\langle \int_{z_i}^{z} \frac{dz'}{z'} \int_0^M dM_r[G(\alpha z') - F(\alpha z')]L_\nu^{(1)}(0,T) \\ \times \left\{ \int_0^M dM_r\left[F(\alpha z') - \left(1 + \frac{C}{Bz^2}\right)G(\alpha z')\right]L_\nu^{(1)}(0,T)\right\}^{-1}\Big\rangle .\end{aligned} \tag{43}$$

If we introduce equation (43) into equation (38), we can also express t as a function of z:

$$\begin{aligned}t(z) = T_i^{-6} \int_{z_i}^{z} \frac{dz'}{z'} \left|\frac{T(z')}{T_i}\right|^{-6} \\ \times \left\{ \int_0^M dM_r\left[\frac{2L_\nu^{(1)}(0,T)}{CT^8F(0)}\right]\left[F(\alpha z') - \left(1 + \frac{C}{Bz'^2}\right)G(\alpha z')\right]\right\}^{-1}.\end{aligned} \tag{44}$$

The integrations involved in equations (43) and (44) can easily be carried out numerically if ρ, Γ, $v(\partial\mu_n/\partial v - \partial\mu_p/\partial v - \partial\mu_e/\partial v)$, and ξ are known at every point in the star. However, the present level of understanding of the strong interactions does not allow us to determine these quantities reliably. As a consequence, the neutrino luminosity of a non-vibrating neutron star is somewhat uncertain, and the uncertainty is considerably more severe when vibrations are considered.

In view of the large uncertainties, we perform the integrations of equations (43) and (44) only for an extremely crude model of a neutron star. We feel that a simple model will serve just as well as a complicated one to illustrate the behavior of the solutions.

We thus consider a star with constant density ρ equal to $2\rho_{\text{nucl}}$ and with a mass equal to 0.64 $M_\odot$. We also assume that $\xi(r) = 1$ everywhere in the star. We treat the neutron gas as a perfect non-relativistic Fermi gas and thus set Γ equal to $\frac{5}{3}$; using equation (5) of Bahcall and Wolf (1965*a*), we have

$$v\left(\frac{\partial\mu_n}{\partial v} - \frac{\partial\mu_p}{\partial v} - \frac{\partial\mu_e}{\partial v}\right) \approx \tfrac{1}{3}E_F(e) \approx 36.5 \text{ MeV} . \tag{45}$$

From equation (45) and equation (37b) we find that $a = 1270 \times 10^9$ ° K. The gravitational binding energy Ω of the star is 0.9×10^{53} ergs; for the case $\xi = 1$ and $\Gamma = \frac{5}{3}$, the quantity B, which is defined by equation (30), is equal to Ω. Equation (26) gives the value 4×10^{29} ergs $(° \mathrm{K})^{-2}$ for C.

For the case of a constant-density star, we can set az_c equal to a constant u_c and write equation (40) in the form

$$u_c^2[F(u_c)/G(u_c) - 1] = Ca^2/B \,. \tag{46}$$

We plot u_c as a function of Ca^2/B in Figure 2. In the present case, Ca^2/B is approximately equal to 7.4, which implies that $u_c \approx 7.2$. The corresponding value of z_c is $5.66 \times 10^{-12} (° \mathrm{K})^{-1}$.

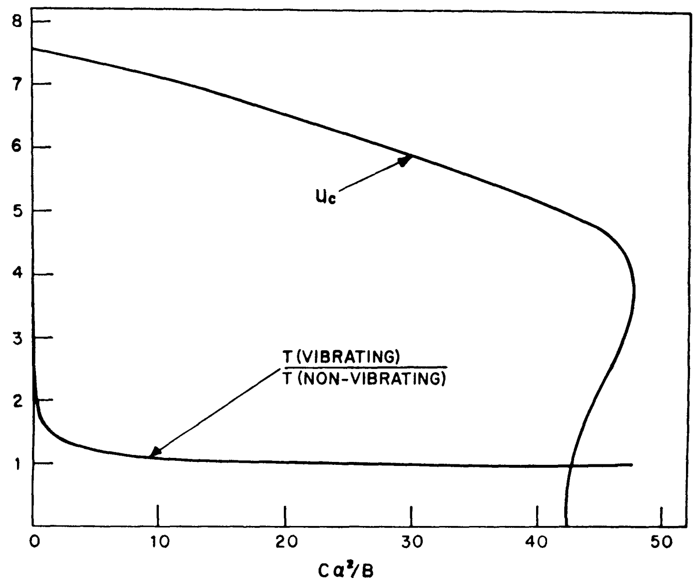

FIG. 2.—Quantities u_c and T(vibrating)/T(non-vibrating) plotted as functions of Ca^2/B. The plot of T(vibrating)/T(non-vibrating) indicates the extent to which vibrations can increase the temperature of a neutron star; the quantity Ca^2/B depends on the details of the structure of the neutron star.

The results of the integrations of equations (43) and (44) are plotted in Figure 3. The solid lines represent the simple solutions corresponding to $u_i = z_i = 0$, $u_i = z_i = \infty$, and $u_i = 7.2$, $z_i = 5.66 \times 10^{-12}$ $(° \mathrm{K})^{-1}$. The value of Bz_c^2/C, the ratio of vibrational energy to thermal energy, is approximately 7 for this solution. Solutions corresponding to a few other values of u_i are represented by dashed lines. We can summarize the content of Figure 3 as follows:

i) For $u_i > 7.2$ the vibrational amplitude falls rapidly at the beginning, while the temperature either rises or, in some cases, falls slowly. The ratio z of vibrational amplitude to temperature rapidly approaches its equilibrium value of 5.66×10^{-12} $(° \mathrm{K})^{-1}$. The vibrational amplitude and temperature always stay between the values given by the analytic formulae for $u_i = \infty$ and $u_i = 7.2$.

ii) For $u_i < 7.2$ the temperature falls rapidly, the vibrational amplitude less rapidly, until the ratio of vibrational amplitude to temperature reaches 5.66×10^{-12} $(° \mathrm{K})^{-1}$. The vibrational amplitude and temperature remain between the values given by the analytic formulae for $u_i = 7.20$ and $u_i = 0$.

It is clear from Figure 3 that the knowledge of the three limiting cases described in § VI*b* gives a good indication of the variation of temperature and vibrational amplitude with time for other initial values of z. In fact, unless the initial vibrational amplitude is extremely small, the solution rapidly approaches the asymptotic solution given by equations (41a) and (41b).

VII. COMPARISON WITH PREVIOUS RESULTS

We want to estimate the effect of vibrations on the rate at which a neutron star cools. In particular, we are interested in cases where the stellar temperature has had time to decrease substantially below its initial value; at such long times, the temperature of a vibrating star can be estimated from equation (41a) and is approximately proportional

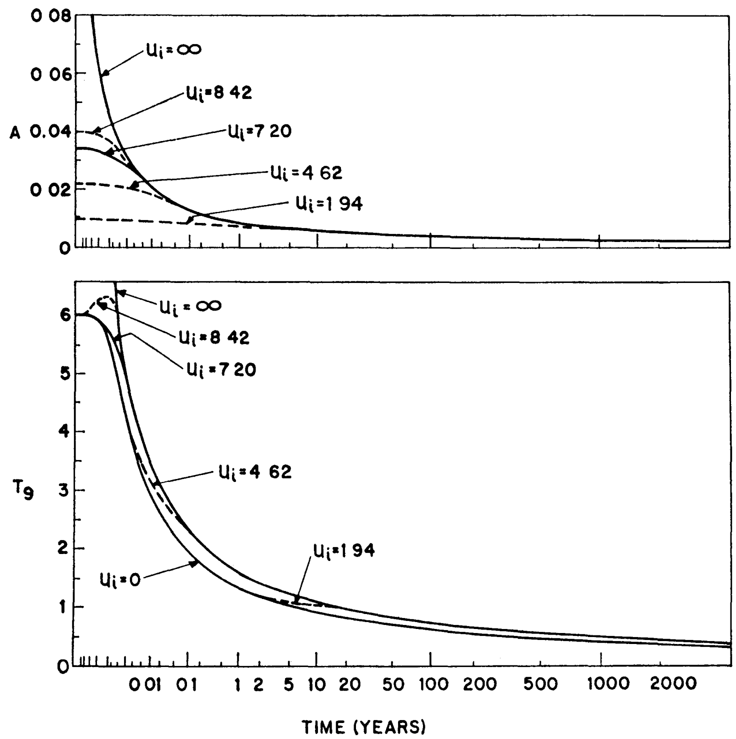

FIG. 3.—Vibrational amplitude A and temperature T as functions of time. The curve marked $u_i = 0$ corresponds to a non-vibrating star. The curve marked $u_i = \infty$ corresponds to a star that started with a very large vibrational amplitude. The calculations were carried out for an idealized neutron star with a constant density of $2\rho_{\text{nucl}}$ and a mass of $0.64\, M_\odot$. Using expression (31) for W_T, taking $\Omega = 0.9 \times 10^{53}$ ergs and $\Gamma = \frac{5}{3}$, the upper plot gives the asymptotic value 3.5×10^{53} ergs year$^{-1/3}$ for the vibrational energy.

to $t^{-1/6}$. Using equation (39) to estimate the temperature of a non-vibrating star at long times, we then obtain

$$\frac{T(\text{vibrating})}{T(\text{non-vibrating})} = \left\{\frac{Bz_c^2}{C}\int_0^M T^{-8}L_\nu^{(1)}(0,T)F(0)dM_r \times \left[\int_0^M T^{-8}L_\nu^{(1)}(0,T)G(u_c)dM_r\right]^{-1}\right\}^{1/6}, \tag{47}$$

where $u_c = \alpha(r)z_c$.

For simplicity, let us consider a star in which α is independent of r. We then have

$$\frac{T(\text{vibrating})}{T(\text{non-vibrating})} = \left[\frac{C\alpha^2}{Bu_c^2}\frac{G(u_c)}{F(0)}\right]^{-1/6}. \tag{48}$$

The value of $u_c(=\alpha z_c)$ can be calculated as a function of $C\alpha^2/B$, using equation (41). The resulting values of u_c and T(vibrating)/T(non-vibrating) are plotted for a constant-density star in Figure 2. Except for the unlikely case where $C\alpha^2/B \ll 1$, we see that T(vibrating) is only slightly greater than T(non-vibrating). For the case considered in § VII*c*, $C\alpha^2/B$ is approximately equal to 7.4, and

$$\frac{T(\text{vibrating})}{T(\text{non-vibrating})} \approx 1.12 \,. \tag{49}$$

The results of Tsuruta and Cameron (1966*a*) indicate that the surface temperature T_e of a neutron star is roughly proportional to the core temperature. Equation (49) and Figure 2 therefore indicate that the photon luminosity, which is proportional to T_e^4, is not changed dramatically by the existence of vibrations. However, it should be pointed out that all the temperature calculations presented so far are based on the assumption that the rate of energy loss by the Urca processes is large compared to the photon luminosity. If this assumption is relaxed, the effect of vibrations can become more dramatic, as we shall see at the end of this section.

We also want to evaluate the effect of temperature on the rate of dissipation of vibrational energy. As we pointed out in § I, a cold, vibrating star could exist in nature if there were some mechanism for carrying heat from the interior to the surface much more efficient than those considered so far. In this case, the ratio z of vibrational amplitude A to temperature T would always be much greater than its equilibrium value z_c, which is given by equation (40). The dimensionless parameter $u(=\alpha z)$ would also be much greater than its equilibrium value, which was found to be about 7.2 for the case studied at the end of § VI.

Figure 1 suggests that the ratio $F(u)/G(u)$ approaches an asymptotic value as u becomes much larger than 7.2. Using equations (11a) and (23), one can show that this asymptotic value is $\frac{3}{8}$. In a cold neutron star, three-eighths of the vibrational energy dissipated by the Urca reactions thus is radiated away in the form of neutrinos, while the balance is converted into thermal energy.

From equation (23) the asymptotic expression for $G(u)$ is found to be

$$G(u) = \frac{u^8}{768} \, ; \tag{50}$$

introducing equation (50) into equation (32), we find for A the differential equation

$$A^{-7}\frac{dA}{dt} = -\frac{1}{384}\int_0^M \frac{L_\nu^{(1)}(0,T)}{BT^8F(0)}\,\alpha^8 dM_r \, , \tag{51}$$

which can be integrated to give

$$A(T\equiv 0) = 2t^{-1/6}\left[\int_0^M \frac{L_\nu^{(1)}(0,T)}{BT^8F(0)}\,\alpha^8 dM_r\right]^{-1/6} . \tag{52}$$

On the other hand, the vibrational amplitude of a hot neutron star is given by equations (41a) and (41b). Taking the ratio of these two results for times long enough that the vibrational amplitude has dropped significantly below its initial value, we obtain

$$\begin{aligned}\frac{A(T\equiv A/z_c)}{A(T\equiv 0)} \approx \Bigg\{ & \frac{1}{768}\int_0^M T_i^{-8}L_\nu^{(1)}(0,T_i)(\alpha z_c)^8 dM_r \\ & \times\left[\int_0^M T_i^{-8}L_\nu^{(1)}(0,T_i)G(\alpha z_c)\,dM_r\right]^{-1}\Bigg\}^{1/6} ,\end{aligned} \tag{53}$$

or, for a star with constant α,

$$\frac{A(T \equiv A/z_c)}{A(T \equiv 0)} \approx \left[\frac{u_c{}^8}{768G(u_c)}\right]^{1/6}. \tag{54}$$

For the numerical example considered in § VI*c*, we then have

$$\frac{A(T \equiv A/z_c)}{A(T \equiv 0)} \approx 0.92\,. \tag{55}$$

The effect of temperature on the vibrational amplitude does not seem to be dramatic.

On the other hand, the surface emission of this hypothetical cold, vibrating neutron star could be significantly greater than that of a hot, vibrating star. Five-eighths of the vibrational energy dissipated by the Urca reactions would eventually be radiated from the surface. Taking the simple uniform-density model considered above and using equations (29) and (52), we find a surface luminosity of 1.3×10^{37} ergs sec^{-1} for a star 300 years old.

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REFERENCES

Bahcall, J. N., and Wolf, R. A. 1965, *Phys. Rev.*, **140**, B1452.
Chau, W.-Y. 1967, *Ap. J.*, **147**, 664.
Chiu, H -Y , and Salpeter, E. E. 1964, *Phys. Rev. Letters*, **12**, 413.
Colgate, S. A., and White, R. H 1966, *Ap. J.*, **143**, 626.
Finzi, A 1964, *Ap. J* , **139**, 1398.
——— 1965*a*, *Phys. Rev.*, **137**, B472.
——— 1965*b*, *Phys. Rev. Letters*, **15**, 599.
Friedman, H. (private communication).
Friedman, H , Byram, E. T , and Chubb, T. A. 1967, *Science*, **156**, 374.
Ginzburg, V. L., and Kirzhnits, D. A. 1965, *Soviet Phys. J. Exper and Theoret. Phys.*, **20**, 1346.
Hansen, C J 1966, unpublished Ph D. thesis.
Hansen, C. J., and Tsuruta, S. 1967, *Canadian J. Phys.*, **45**, 2823.
Ledoux, P., and Walraven, T. 1958, *Hdb. d. Phys.*, **51**, 353.
Meltzer, D. W , and Thorne, K S. 1967, *Ap. J.*, **145**, 514.
Ruderman, M.A , and Festa, G. 1966 (private communication).
Shklovskii, I S. 1966, *Soviet Astr.—AJ*, **10**, 6.
Tsuruta, S , and Cameron, A. G. W. 1966*a*, *Canadian J. Phys* , **44**, 1863.
———. 1966*b*, *ibid.*, p. 1895
Wolf, R A 1966, *Ap J* , **145**, 834.
Woltjer, L. 1964, *Ap. J.*, **140**, 1309.